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I. R. Goodman, and G. F. Kramer

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San Diego, CA 92152–5001

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Relational Event Algebra (REA) is a new mathematical tool which provides an explicit algebraic reconstruction of events (appropriately designated as relational events) when initially only the formal probability values of such events are given as functions of known contributing event probabilities. In turn, once such relational events are obtained, one can then determine the probability of any finite logical combination, and in particular, various probabilistic distance measures among the events. A basic application of REA is to test hypotheses for the similarity of distinct models attempting to describe the same events such as in data fusion and combination of evidence.

This paper considers new motivation for the use of REA, as well as a more general decision-making framework where system performance and redundancy / consistency tradeoffs are considered.

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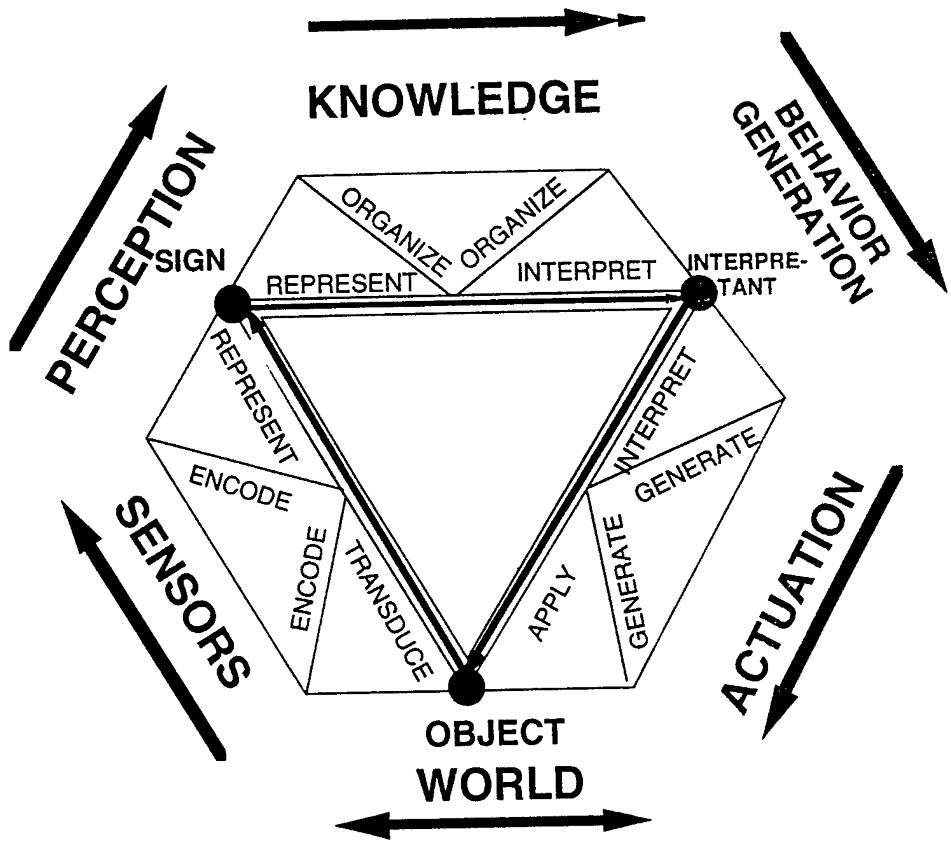

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EXTENSION OF RELATIONAL EVENT ALGEBRA TO A GENERAL DECISION MAKING SETTING

I.R. Goodman and G.F. Kramer *

**NCCOSC RDTE DIV (NRAD) 4221,
SEASIDE, BLDG 600, RM 341A,
53118 GATCHELL RD,
SAN DIEGO, CA 923152-7446**

Abstract

Relational Event Algebra (REA) is a new mathematical tool which provides an explicit algebraic reconstruction of events (appropriately designated as relational events) when initially only the formal probability values of such events are given as functions of known contributing event probabilities. In turn, once such relational events are obtained, one can then determine the probability of any finite logical combination, and in particular, various probabilistic distance measures among the events. A basic application of REA is to test hypotheses for the similarity of distinct models attempting to describe the same events such as in data fusion and combination of evidence.

This paper considers new motivation for the use of REA, as well as a more general decision-making framework where system performance and redundancy / consistency tradeoffs are considered.

1 INTRODUCTION: BASIC ISSUES

Two fundamental quests in intelligence gathering are:

(i) The determination whether a pair of information / data models are consistent with respect to each other or are significantly different

(ii) If the models relative to (i) are considered similar, how to combine them.

This paper will be concerned with issue (i). Future work will deal with the equally important item (ii). The purpose of this paper is to motivate more interest in this interesting and fundamental area, rather than to present detailed results. (Space limitations here preclude this anyway.)

At the outset, let us agree that the naive measure of distance between any two given events a,b, in some appropriately chosen boolean or sigma algebra B relative to a probability space (Ω,B,P), is

$$n_P(a,b) = |P(a)-P(b)| . \quad (0)$$

This is a legitimate distance function between a and b, but it is not satisfactory: One can find in general events a and b quite distinct --even disjoint -- for which their probabilities are similar. However, if the conjunction ab is also known or P(ab) is otherwise obtainable, then one natural measure of association is given by the *standard probabilistic distance* d_P, where

$$d_P(a,b)=P(a+b) = P(a'b) + P(ab') = P(a) + P(b) - 2P(ab), \quad (1)$$

where the + inside the P( ) operator is the usual boolean symmetric difference operator. It is well known [1] that this function supplies a pseudometric -- and indeed, a metric, modulo zero probability measure events. In fact it can also be shown (via use of the relative atom probabilities to verify the triangle inequality) that the *relative probabilistic distance*

$$r_P(a,b) = P(a+b)/P(a\vee b) \quad (2)$$

-- a possibly more satisfactory measure of association -- is also a pseudometric. Still another natural candidate for measuring probabilistic closeness is provided by a symmetrization of conditional probabilities

$$\begin{aligned} e_P(a,b) &= -(1/2)\log(P(a|b)P(b|a)) \\ &= (1/2)\log P(a) + (1/2)\log P(b) - \log P(ab). \quad (3)\end{aligned}$$

One can inquire as to what are the basic desirable properties that probabilistic measures of similarity should possess. Table 1 illustrates this for the four measures of similarity discussed above. It should be pointed out that the only proposed similarity measure of the four discussed here that satisfies the desirable properties 2-4 and does not satisfy the undesirable ones 5-9 is r_P.

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Property	Similarity Measure m_P			
	n_P	d_P	r_P	e_P
1 $m_P(a,b)$ function only of P(a),P(b)	Y	N	N	N
2 $m_P(a,b)$ function only of P(a),P(b), P(ab)	N	Y	Y	Y
3 $m_P(a,b)$ is a (pseudo)metric	Y	Y	Y	N
4 $m_P(a,b)$ monotone decreasing in P(ab) for fixed P(a),P(b)	N	Y	Y	Y
5 There exists a,b such that $m_P(a,b)$ small, P(a) large, P(b) small	N	N	N	Y
6 There exists a,b such that $m_P(a,b)$ small, P(ab) small	Y	Y	N	N
7 There exists a,b such that $m_P(a,b)$ small, P(a),P(b) large, P(ab) small	N	N	N	N
8 There exists a,b such that $m_P(a,b)$ small, P(a),P(b) medium, P(ab) very small	Y	N	N	N
9 If P(a),P(b) small, then $m_P(a,b)$ small	Y	Y	N	N

Table 1. Comparison of Properties of Some Measures of Event Similarity

Basic research issues connected with Table 1 include:

1. Characterize the class of similarity measures satisfying a particular property or properties, such as 1 or 2.

2. What properties, in addition to 2, guarantee uniqueness ?

3. What other properties distinguish between n_P and d_P, r_P, e_P?

4. What are the connections between similarity measures proposed here and pattern recognition techniques ?

2. SIMILARITY OF MODELS VIA FORMAL TESTS OF HYPOTHESES USING PROBABILISTIC DISTANCES

By making a natural second order probability assumption concerning the uniformity of distribution of the atomic probabilities P(ab), P(ab'), P(a'b), for formally varying P, one can test hypotheses concerning significance of association for each of the similarity measures discussed in Section 1.1 and others as well [2]. More specifically, letting null hypothesis H_0 correspond to $a \neq b$ and H_1 to the alternative $a = b$ (equivalently, up to zero probability measure, P(a'b)=P(ab')=0), under the above joint uniformity assumption we can show that the cdf (cumulative distribution function) F_1 of statistic $d_P(a,b)$ is

$$F_1(t) = t^2 \cdot (3-2t)\ , \text{ all } 0 \le t \le 1 \qquad (4)$$

while the cdf F_2 for statistic $r_P(a,b)$ is

$$F_2(t) = t^2, \text{ all } 0 \le t \le 1. \qquad (5)$$

Thus, e.g., when $s = r_P(a,b)$ is computed numerically, the significance level

$$\begin{aligned}\gamma &= P(\text{reject } H_0 \mid H_0 \text{ true})\\ &= P(\text{accept } H_1 \mid H_0 \text{ true})\\ &= F_2(s)\end{aligned} \qquad (6)$$

means as usual that for all tests of hypotheses H_0 vs. H_1 using the given s with any chosen significance level α (other than γ) one should:

$$\begin{aligned}&\text{accept } H_0 \text{ (and reject } H_1) \text{ , if } \alpha < \gamma ,\\ &\text{reject } H_0 \text{ (and accept } H_1), \text{ if } \alpha > \gamma.\end{aligned} \qquad (7)$$

Equivalently, for any pre-chosen significance level α,

$$\begin{aligned}&\text{accept } H_0 \text{ , if } s > C_\alpha ,\\ &\text{reject } H_0 \text{ , if } s < C_\alpha ,\end{aligned} \qquad (8)$$

where threshold constant C_α is determined from

$$F_2(C_\alpha) = \alpha\ , \qquad (9)$$

i.e.,

$$C_\alpha = \alpha^{1/2}\ . \qquad (10)$$

(Again, see [2] for more details.)

3. THE SITUATION WHEN PROBABILISTIC CONJUNCTIONS ARE NOT AVAILABLE

When events a,b above are such that the actual conjunction ab -- or at least the evaluation P(ab) -- is *not* known, none of the above natural procedures can be carried out since both types of probabilistic distance functions depend explicitly on P(ab) (as well as the known marginal probabilities P(a), P(b)). Recall the usual tightest (Fréchet-Hailperin) bounds to P(ab) and P(avb) [3]:

$$\max(P(a) + P(b) - 1, 0) \le P(ab) \le \min(P(a),P(b))$$

$\leq \; wP(a) + (1-w)P(b)$

$\leq \; \max(P(a),P(b)) \; \leq P(a \vee b) \leq \min(P(a) + P(b), 1)$ (11)

for any constant w, $0 \leq w \leq 1$. Applications of eq.(11) to $d_P(a,b)$ and/or $r_P(a,b)$, in terms of knowing only the marginal probabilities P(a),P(b), show in general a wide leeway for error relative to the actual computations [2]:

$$|P(a)-P(b)| \leq d_P(a,b) \leq \min(P(a)+P(b), 2-P(a)-P(b)), \quad (12)$$

$$1-\min(P(a)/P(b), P(b)/P(a)) \leq r_P(a,b) \leq \min(1, 2-P(a)-P(b)).$$

4. SOME EXAMPLES LEADING TO MODEL COMPARISONS FOR SIMILARITY

Consider the estimation of the probability of enemy attack tomorrow at the shore from two different experts who take into account the contributing probabilities of good weather holding, a calm sea state, and the enemy having an adequate supply of type I weapons. In the simplest type of modeling the experts may provide their respective probabilities as weighted sums of the contributing probabilities:

Example 1.

$$\text{Model 1: } P(a) = w_{11}P(c) + w_{12}P(d) + w_{13}P(e)$$
$$(13)$$
$$\text{Model 2: } P(b) = w_{21}P(c) + w_{22}P(d) + w_{23}P(e),$$

where

$0 \leq w_{ij} \leq 1$, $w_{i1} + w_{i2} + w_{i3} = 1$, i=1,2,

a = enemy attacks tomorrow, according to expert 1,
b = enemy attacks tomorrow, according to expert 2,
c = good weather will hold,
d = calm sea state will hold,
e = enemy has adequate supply of type 1 weapons. (14)

Note that the simple-appearing linear forms in eq.(1) may well belie a more complex situation. Unless the events c,d,e are disjoint and exhaustive with respect to a and b, the two models in eq.(13) do not correspond to the usual total probability expansion forms where, e.g. $w_{11} = P(a|c)$, $w_{12} = P(a|d)$, etc. Furthermore, each expert may weight quite differently the contributions of c,d, or e to enemy attack. Thus, the fundamental issue here is: Are both experts' models similar enough to be combined into a single model or are they significantly distinct as to be either held separate until further information arrives or one or both are to be discarded?

Relational event algebra (REA) has been developed to address the above and related issues. (See, e.g., [2] for background.) By appropriately imbedding the events c,d,e in a higher order boolean algebra and utilizing a corresponding higher order probability space a reasonable solution to the problem can be obtained. Thus, letting $(\Omega,\mathcal{B},P)$ be the given probability space associated with c,d,e (in boolean or sigma algebra $\mathcal{B}$), the higher order probability space $(\Omega_o,\mathcal{B}_o,P_o)$ is essentially the product probability space formed out of a countable infinity of factor spaces, each identical to $(\Omega,\mathcal{B},P)$. Then, it can be shown that one can obtain ([2], Theorem 3 for n=3 in conjunction with eqs.(92),(93)) the *relational event*

$$\begin{aligned} a = \; & (cde \times \Omega_o) \vee (cde' \times \theta(w_{11} + w_{12})) \\ & \vee (cd'e \times \theta(w_{11}+w_{13})) \vee (c'de \times \theta(w_{12} + w_{13})) \\ & \vee (cd'e' \times \theta(w_{11})) \vee (c'de' \times \theta(w_{12})) \\ & \vee (c'd'e \times \theta(w_{13})), \end{aligned} \quad (15)$$

where

$$\Omega_o = \Omega \times \Omega \times \Omega \times ... \quad (16)$$

and the $\theta(\,)$ are certain *(constant) conditional events* in $\mathcal{B}_o$ (see Example 4, Section 5 for explanation) such that for essentially all choices of probability measure P, the compatibility condition with eq.(13) holds, formally replacing P by P_o on the left-hand side:

$$P_o(a) = w_{11}P(c) + w_{12}P(d) + w_{13}P(e). \quad (17)$$

A similar result holds for b. Then, it can be shown that the conjunction probability here is

$$\begin{aligned} P_o(ab) = P(cde) & + P(cde')\min(w_{11}+w_{12},w_{21}+w_{22}) \\ & + P(cd'e)\min(w_{11}+w_{13},w_{21}+w_{23}) \\ & + P(c'de)\min(w_{12}+w_{13},w_{22}+w_{23}) \\ & + P(cd'e')\min(w_{11},w_{21}) \\ & + P(c'de')\min(w_{12},w_{22}) \\ & + P(c'd'e)\min(w_{13},w_{23}), \end{aligned} \quad (18)$$

and

$$P_o(a \vee b) = P(a) + P(b) - P_o(ab); \quad (19)$$

or equivalently, replacing min by max throughout eq.(18). Next, eqs.(18), (19) in conjunction with eqs.(1),(2) can be used to compute the similarity measures $d_P(a,b)$ or $r_P(a,b)$ here. In turn, these computations can be applied as outlined in Section 2 to test hypotheses about the distinctness or redundancy of the two models in eq.(13).

An even more complicated problem arises when one or both experts produce nonlinear models for their respective opinion probabilities such as the exponential forms:

Example 2.

$$\text{Model 1: } P(a) = w_{11}(P(c))^{r1} + w_{12}(P(d))^{r2} + w_{13}(P(e))^{r3}$$
$$(20)$$
$$\text{Model 2: } P(b) = w_{21}(P(c))^{s1} + w_{22}(P(d))^{s2} + w_{23}(P(e))^{s3},$$

where a,b,c,d,e and the w_{ij} are as in eq.(14) and the s_{ij} are any fixed positive real numbers. Still another example can arise from natural language and fuzzy logic considerations. In this case, two experts may be responding initially via linguistic narratives describing independently the same target from different perspectives or using different sensor systems:

Example 3.

Model 1: Ship A is very long and has a large number of Q-type weapons on deck

(21)

Model 2: Ship A is fairly long or it is actually two or more different ships

Translating the models in eq.(21) to fuzzy logic terminology (see, e.g. [2] for background) yields

$$\text{Model 1: } t(a) = \min((f_{long}(\text{lngth}(A)))^2, f_{large}(\#(Q))) \qquad (22)$$

$$\text{Model 2: } t(b) = \max((f_{long}(\text{lngth}(A)))^{1.5}, f_{2+}(\#(R))),$$

where

t(a) = truth or possibility of the description of ship A using Model 1,
t(b) = truth or possibility of the description of ship A using Model 2;

f_{long}:Pos.Reals $\to$ [0,1],

f_{large}:Pos.Reals $\to$ [0,1] ,

f_{2+} : {0,1,2,3,...) $\to$ [0,1]

(23)

are appropriately determined fuzzy set membership functions representing "long", "large", "2 or more", respectively;

A = Ship A,
Q = Q-type weapons on A
R = Ship A actually consists of set R of different ships;

(24)

and measurement functions

lngth( ) = length of ( ) in feet,
#( ) = no. of ( ).

(25)

The one-point coverage relation between fuzzy logic and random sets (see, e.g. Goodman [4]) is

$$f_B(x) = P(x \text{ in } S(f_B)) \text{ , all x in D ,} \qquad (26)$$

where f_B:D $\to$ [0,1] is a fuzzy set membership function corresponding to attribute B and random set $S(f_B) \subseteq D$ need not be unique in general. This produces the following corresponding probability evaluations in eq.(22):

$$\text{Model 1: } P(a) = \min((P(c))^2, P(d)),$$
$$\text{Model 2: } P(b) = \max((P(c))^{1.5}, P(e)), \qquad (27)$$

where now

a = description of ship A via Model 1 is valid,
b = description of ship A via Model 2 is valid,
c = lngth(A) in $S(f_{long})$,
d = #(Q) in $S(f_{large})$,
e = #(R) in $S(f_{2+})$. (28)

As in Example 1, REA can be used to address Examples 2 and 3, but again, lack of space here precludes any details. (See [2] for related results.)

5. THE CONDITIONAL EVENT ALGEBRA PROBLEM.

Consider yet another example which also illustrates a special important case of the REA problem. Here, the functions of the contributing probabilities are all simple divisions yielding conditional probabilities: Determine the consistency of conditional probabilities in eq.(29), utilizing, if possible, the distance metrics discussed in the Introduction:

Example 4.

$$\text{Model 1: } P(c|d) = P(cd)/P(d),$$
$$\text{Model 2: } P(c|e) = P(ce)/P(e). \qquad (29)$$

These values could represent, e.g., two posterior descriptions of parameter c via different data sources corresponding to events d,e.

Until recently there was no standard way to deal with the natural similarity issues connected with eq.(29). Again, this is because until only the last few years, there was no sound approach to determining just what the conjunction of the relational events (c|d)&(c|e) means – using obvious notation here to denote the relational (actually, conditional) events a = (c|d), b = (c|e), corresponding to given conditional probabilities P(c|d), P(c|e), respectively. This also applies to determining $d_P((c|d),(ce))$ and $r_P((c|d),(c|e))$. Commencing with the work of Calabrese in 1987 [5] based upon his earlier ideas and that of Adams [6], interest in the conditional event problem

was significantly revived. (Actually, the problem had been considered previously from time to time by various individuals beginning with Boole [7], made more rigorous by Hailperin [8] over a hundred years later. In addition there was the independent work of DeFinetti [9] and Schay [10], among others. But, little follow-up work and cohesion of effort was accomplished. See Goodman [11] for a more thorough early history of the problem.)

Following Calabrese' contributions (see also the more recent work in [12]), Goodman and Nguyen addressed the conditional event algebra problem, taking a different approach from Calabrese - Adams [13,14]. However, all of these proposed CEA's had drawbacks, the paramount one being that none of the proposals yielded algebraic / logical structures which take a boolean or sigma algebra form, necessary in order to be able to apply the already richly developed body of standard probability results, including expansion forms, limit theorems, inequalities, and natural sample space and frequency interpretations. But, still more recently, a number of these difficulties -- including the boolean algebraic structure deficiency -- have been overcome with the introduction and extension of a CEA by Goodman and Nguyen [15,16] which was earlier independently proposed by Van Fraasen [17] and by McGee [18], but neglected. This CEA is based essentially on the algebraic analogue of the standard expansion of arithmetic division in terms of an infinite series (of sums of power terms) and has a number of appealing natural properties, including a full independence-based characterization [15]. For purpose of reference, we denote it as the product space CEA.

It is interesting to note that despite the recent progress made in developing CEA's, including the publication of an entire issue of the *IEEE Transactions on Systems, Man & Cybernetics* [19], two monographs devoted to the subject [20,21], as well as a host of workshops and papers, a goodly number of researchers in rule-based systems are still unaware of the potential use of CEA in determining the similarity of rules. This also holds for the related problem of deletion or addition of rules to a given rule-based system based on the degree of mutual distinctness. For example, Rowe ([22], especially Chapter 8) explicitly recognizes the importance of both assigning conditional probabilities to inference rules as natural measures of their overall uncertainty and determining exactly - or at least via bounds - probabilities of the basic logical (&,v) combinations of such rules. However, Rowe concludes:

"The last section shows that combining probabilities in rule-based systems is very important. Surprisingly, there is no fully general mathematical approach to combining: the problem can be proved mathematically intractable. But, we can give some formulas that hold under particular assumptions.."

Needless to say, the mathematical development of the product space CEA contradicts Rowe's claim of intractability. Furthermore, his "conservative" and "liberal" bounds on the probability evaluation of logical combinations of rules (which he tacitly assume are conditional events in a boolean algebra to be able to obtain bounds -- though he does not display the conditional events explicitly), as well as his independence assumptions where invoked, correspond completely to our previously derived results as displayed, e.g., in [15]. In particular, see the Fréchet-Hailperin bounds in eq.(11) where events a and b represent conditional events such as (c|d) and (c|e) in eq.(24). Because of space limitations, we can only present a few basic results for the product space CEA as applicable to eq.(29):

Again, denote the given probability space for events c,d,e,.. as (Ω,B,P) (with c,d,e,.. in boolean or sigma algebra B, etc.). Then,

$$(c|d) = (cd \times \Omega_o)\ \vee\ (d' \times cd \times \Omega_o)\ \vee\ (d' \times d' \times cd \times \Omega_o)\ \vee \ldots, \qquad (30)$$

where

$$\Omega_o = \Omega \times \Omega \times \Omega \times \ldots, \qquad (31)$$

(c|d) satisfies the basic compatibility relation for eq.(29), again formally replacing P by P_o in the left-hand side,

$$P_o((c|d)) = P(c|d), \qquad (32)$$

for essentially all choices of P.

(c|e) has a similar expansion with d replaced by e, and noting that all of these conditional events lie in the boolean algebra B_o generated by $B \times B \times B \times \ldots$, corresponding to product probability space (Ω_o,B_o,P_o). Despite the countable infinite number of factors in the above structures, the basic logic of boolean operations among these conditional events yields finite computational results such as

$$P_o((c|d)\&(c|e)) = (P(cde) + (P(cde')P(c|e)) + (P(cd'e)P(c|d))) / P(d \vee e) \qquad (33)$$

and

$$P_o((c|d) \vee (c|e)) = P(c|d) + P(c|e) - P_o((c|d)\&(c|e)) \qquad (34)$$

because, again, all of the usual laws of probability apply to this space as well. Then, eqs.(33) or (34) can be used in conjunction with eqs.(1) or (2) (where again, a=(c|d) , b =(c|e)) to obtain the measures of similarity, $d_P((c|d),(c|e))$ or $r_P((c|d),(c|e))$, between the conditional expressions in eq.(29), which, in turn, can be used to test hypotheses as outlined in Section 2.

Extensive numerical experiments are being conducted for examples similar to the four given above; preliminary results show reasonable results and significant improvement compared to the situation where the conjunction or disjunction probabilities are not known and only the Frechét-Hailperin bounds can be utilized (in effect, similar to Rowe's suggestions) [23].

6. THE GENERAL RELATIONAL EVENT PROBLEM

Examples 1-4 above give rise to the following problem, stated for two models, but readily generated to any number of ones: Given probability space (Ω,B,P) with c,d,e,.. in B, and given the models

$$\begin{aligned}&\text{Model 1: } P(a) = f(P(c),P(d),..,P(e)),\\ &\text{Model 2: } P(b) = g(P(c),P(d),...,P(e)),\end{aligned} \quad (35)$$

find a larger probability space (Ω_1,B_1,P_1) and an event-valued function h such that events a = h(c,d,..,e,...;f) and b = h(c,d,..,e,...;g) are in B_1, compatible with eq.(35) in the sense that for essentially all choices of P, the equation holds where P in the left-hand side is replaced by P_1. Once such relational events are constructed, then as shown for Examples 1 and 4, measures of association between the models and corresponding tests of similarity / redundancy can be established.

A natural extension of this is to consider a general decision theory setting where T is a given collection of models or even inference rules and q is a new model or inference rule to be considered for addition to T. Let J(T) be a performance or cost measure associated with T; similarly for $J(T\cup q)$. J(T) could represent the cost in dollars of operating system T or the efficiency of T (such as in kill-probabilities or timeliness of execution of decisions, etc.). Also, determine K(T), K(T,q), where K(T) is the overall measure of similarity within T -- choosing either d_P or r_P, etc. -- averaged over all pairs of elements (i.e., models or inference rules) in T and where K(T,q) is a similarly averaged measure of similarity between K and q. We then seek tradeoffs between J(T) and K(T) and between $J(T\cup q)$, and K(T,q), etc. The idea here is to introduce distances between events in S that in the past were not considered as bona fide, compatible with natural probability evaluations, and therefore did not permit distances or measure of redundancy to be obtained -- such as the special case of S being a set of inference rules. One can then also seek the usual decision-theoretic properties, including game values, minimax and bayes decision functions, and least favorable priors, among others. A basic start toward this goal is presented in [16].

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